Supplemental Document: Anisotropic Spherical Gaussians

Kun Xu[1] Wei-Lun Sun[1] Zhao Dong[2] Dan-Yong Zhao[1] Run-Dong Wu[1] Shi-Min Hu[1]
[1]Tsinghua University, Beijing [2] Program of Computer Graphics, Cornell University

This supplemental document provides additional evaluations and comparisons for the paper entitled *Anisotropic Spherical Gaussians*.

1 Approximating an SG using an ASG with equal bandwidths

The traditionally used Spherical Gaussian (i.e. the von Mises-Fisher distribution), is defined as $G_{iso}(\mathbf{v};\mathbf{p},\nu) = exp(2\nu(\mathbf{v}\cdot\mathbf{p}-1))$, where $\mathbf{p}$ is the lobe direction, and ν is the bandwidth. In practice, it can be approximated by an ASG with equal bandwidths: $G_{iso}(\mathbf{v};\mathbf{p},\nu) \approx G(\mathbf{v};[\mathbf{x},\mathbf{y},\mathbf{p}],[\nu,\nu])$, where $\mathbf{x}$, $\mathbf{y}$ are two arbitrary directions that form a local frame with $\mathbf{p}$. To evaluate its accuracy, we compare our approximation (an ASG with equal bandwidths) to the ground truth (a von Mises-Fisher distribution) in Fig. 1. Note that in all configurations, our approximations match the reference very well.

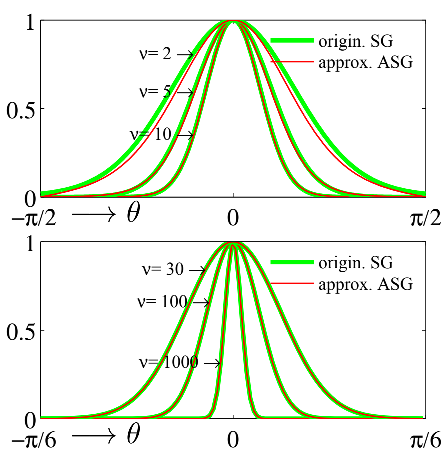


Figure 1: *Comparison of ASGs with equal bandwidths to SGs. θ denotes the angle between direction $\mathbf{v}$ to lobe direction $\mathbf{p}$.*

2 Proof of Eq. 5

Below, we will give the derivations of the analytic solution of the integral $\int_{\theta=0}^{\frac{\pi}{2}} e^{-k\sin^2\theta}\sin\theta\cos\theta\,\mathrm{d}\theta$, which is used in Eq.5 in the paper.

$$\int_{\theta=0}^{\frac{\pi}{2}} e^{-k\sin^2\theta}\sin\theta\cos\theta\,\mathrm{d}\theta \tag{1}$$

$$=\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{k(1-\cos2\theta)}{2}}\sin2\theta\,\mathrm{d}\theta \tag{2}$$

$$\longrightarrow \quad \text{substituting } \theta' = 2\theta$$

$$=\frac{1}{4}\int_{\theta'=0}^{\pi} e^{-\frac{k(1-\cos\theta')}{2}}\,\mathrm{d}\cos\theta' \tag{3}$$

$$\longrightarrow \quad \text{substituting } x = \cos\theta'$$

$$=\frac{1}{4}\int_{x=-1}^{1} e^{-\frac{k(1-x)}{2}}\,\mathrm{d}x \tag{4}$$

$$=\frac{1}{2k}\,e^{\frac{k(x-1)}{2}}\Big|_{-1}^{1} \tag{5}$$

$$=\frac{1}{2k}(1-e^{-k}) \tag{6}$$

3 Derivation of Eq. 8

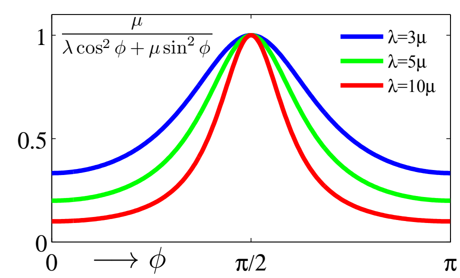


Figure 2: *Plot of $\frac{\mu}{\lambda\cos^2\phi+\mu\sin^2\phi}$.*

By observing the shape of the function $1/(\lambda\cos^2\phi+\mu\sin^2\phi)$ in Fig. 2, we can find that it is also a lobe shape and hence can be reasonably approximated by a circular Gaussian (assuming $\lambda \geq \mu$):

$$\frac{1}{\lambda\cos^2\phi+\mu\sin^2\phi} \approx k_1 + k_2 e^{-k_3\cos^2\phi} \tag{7}$$

where k_1, k_2, k_3 are three parameters to determine. Since the value of $1/(\lambda\cos^2\phi+\mu\sin^2\phi)$ ranges in $[1/\lambda, 1/\mu]$, we set $k_1 = 1/\lambda$, $k_2 = (1/\mu - 1/\lambda)$ to preserve its ranges. k_3 is determined by preserving the second order derivative at the peak position $\phi = \pi/2$, which results in $k_3 = \frac{\lambda-\mu}{\mu}$.

4 Rational approximation of 1D function F in Eq. 9

The 1D function $\mathrm{F}(a)$ is defined as $\mathrm{F}(a) = \int_0^{2\pi} e^{-a\cos^2\phi\,\mathrm{d}\phi}$, which can be approximated by the square root of a rational function:

$$\mathrm{F}(a) \approx \sqrt{\frac{p_1a^3+p_2a^2+p_3a+p_4}{a^4+q_1a^3+q_2a^2+q_3a+q_4}} \tag{8}$$

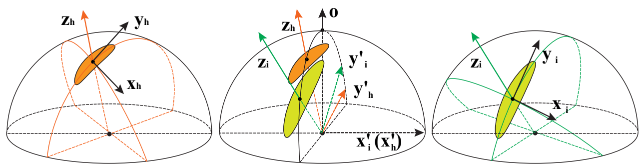


Figure 3: *Spherical Warping.*

where $p_1 = 0.7846, p_2 = 3.185, p_3 = 8.775, p_4 = 51.51, q_1 = 0.2126, q_2 = 0.808, q_3 = 1.523, q_4 = 1.305$. The SSE error is smaller than 4.0×10^{-5}.

5 More evaluations on ASG operators

Product of two ASGs. In Fig. 5, we give more examples on ASG product approximations. In each example, we provide the two input ASGs (the 1st,2nd or the 5th,6th columns), our approximated product represented by an ASG (the 3rd or the 7th columns), and the ground truth product (the 4th or the 8th columns). Note that our approximated results are indistinguishable from the ground truth in all examples.

Convolution of an ASG and an SG. In Fig. 6, we give more examples on ASG convolution approximations. In each example, we provide the input ASG (the 1st or the 5th columns), the input SG (the 2nd or the 6th columns), our approximated convolution represented by an ASG (the 3rd or the 7th columns), and the ground truth convolution (the 4th or the 8th columns). Note that our approximated results are indistinguishable from the ground truth in all examples.

6 Second Derivatives in Eq. 25

As shown in Fig. 3, at direction $\mathbf{h} = \mathbf{z}_h$, we also define a default local frame $[\mathbf{x}'_h, \mathbf{y}'_h, \mathbf{z}_h]$, making the tangent direction $\mathbf{x}'_h$ perpendicular to view direction $\mathbf{o}$ (i.e. $\mathbf{x}'_h = \mathbf{x}'_i$). The three second derivatives of the exponential order $g(\mathbf{i})$ at peak position $\mathbf{i} = \mathbf{z}_i$ are computed as:

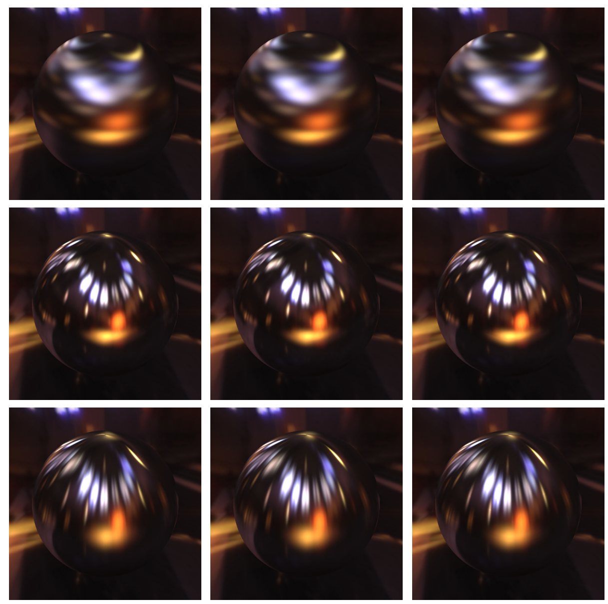
$$\frac{\partial^2 g}{\partial \mathbf{x}'^2_i} = \frac{\lambda(\mathbf{x}_h \cdot \mathbf{x}'_h)^2 + \mu(\mathbf{y}_h \cdot \mathbf{x}'_h)^2}{4(\mathbf{o} \cdot \mathbf{z}_h)^2} \tag{9}$$

$$\frac{\partial^2 g}{\partial \mathbf{x}'_i \partial \mathbf{y}'_i} = \frac{(\mu - \lambda) \cdot (\mathbf{x}_h \cdot \mathbf{x}'_h) \cdot (\mathbf{y}_h \cdot \mathbf{x}'_h)}{4(\mathbf{o} \cdot \mathbf{z}_h)} \tag{10}$$

$$\frac{\partial^2 g}{\partial \mathbf{y}'^2_i} = \frac{\mu(\mathbf{x}_h \cdot \mathbf{x}'_h)^2 + \lambda(\mathbf{y}_h \cdot \mathbf{x}'_h)^2}{4} \tag{11}$$

7 Warping Comparison

Given a view direction $\mathbf{o}$, in order to obtain the corresponding 2D BRDF slice to integrate with incident lighting and visibility functions, we have approximated a warped NDF (represented by ASGs) again using ASGs. In Fig. 4, we evaluate this approximation using Ashikmin BRDF with different anisotropy ratio. As shown in the figure, regardless of anisotropy ratio, the differences between our rendered images with warping approximation and the reference images are subtle.



with warp | without warp | reference

Figure 4: *Warping Comparison of rendered spheres with Ashikmin BRDF model. First row: $n_u = 88.9, n_v = 900$ (anisotropy ratio: 3); second row: $n_u = 20000, n_v = 800$ (anisotropy ratio: 5); third row: $n_u = 20000, n_v = 200$ (anisotropy ratio: 10). The first and second columns give rendered results with and without warping approximations, respectively.*

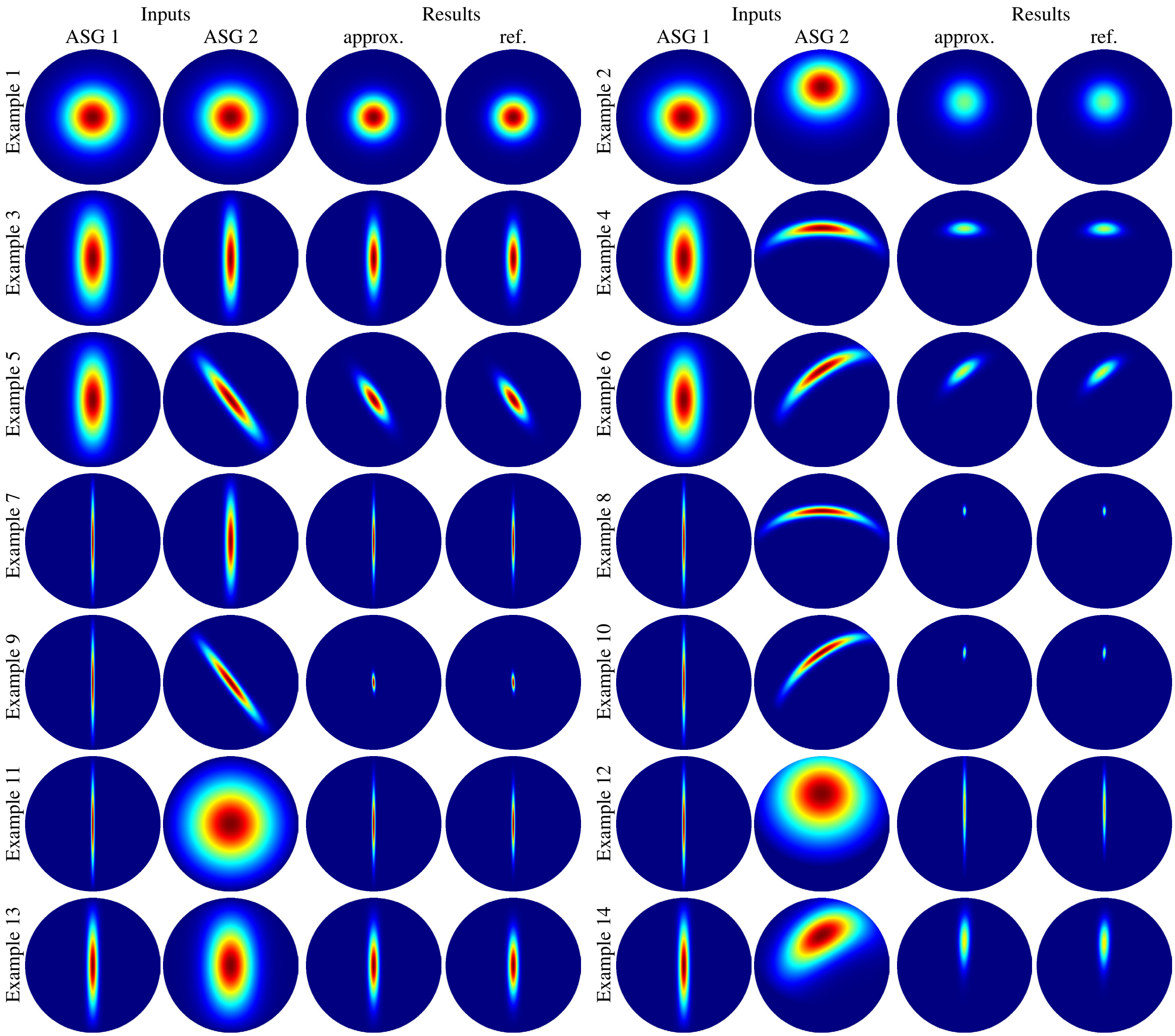


Figure 5: *More Evaluations on ASG product operator. The bandwidth parameters in all examples are listed below. Example 1:* $\lambda_1 = 3$, $\mu_1 = 3$, $\lambda_2 = 3$, $\mu_2 = 3$. *Example 2:* $\lambda_1 = 3$, $\mu_1 = 3$, $\lambda_2 = 3$, $\mu_2 = 3$. *Example 3:* $\lambda_1 = 10$, $\mu_1 = 1$, $\lambda_2 = 50$, $\mu_2 = 1$. *Example 4:* $\lambda_1 = 10$, $\mu_1 = 1$, $\lambda_2 = 50$, $\mu_2 = 1$. *Example 5:* $\lambda_1 = 10$, $\mu_1 = 1$, $\lambda_2 = 50$, $\mu_2 = 1$. *Example 6:* $\lambda_1 = 10$, $\mu_1 = 1$, $\lambda_2 = 50$, $\mu_2 = 1$. *Example 7:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 100$, $\mu_2 = 1$. *Example 8:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 100$, $\mu_2 = 1$. *Example 9:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 100$, $\mu_2 = 1$. *Example 10:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 100$, $\mu_2 = 1$. *Example 11:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 1$. *Example 12:* $\lambda_1 = 1000$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 1$. *Example 13:* $\lambda_1 = 100$, $\mu_1 = 1$, $\lambda_2 = 5$, $\mu_2 = 1$. *Example 14:* $\lambda_1 = 100$, $\mu_1 = 1$, $\lambda_2 = 5$, $\mu_2 = 1$.

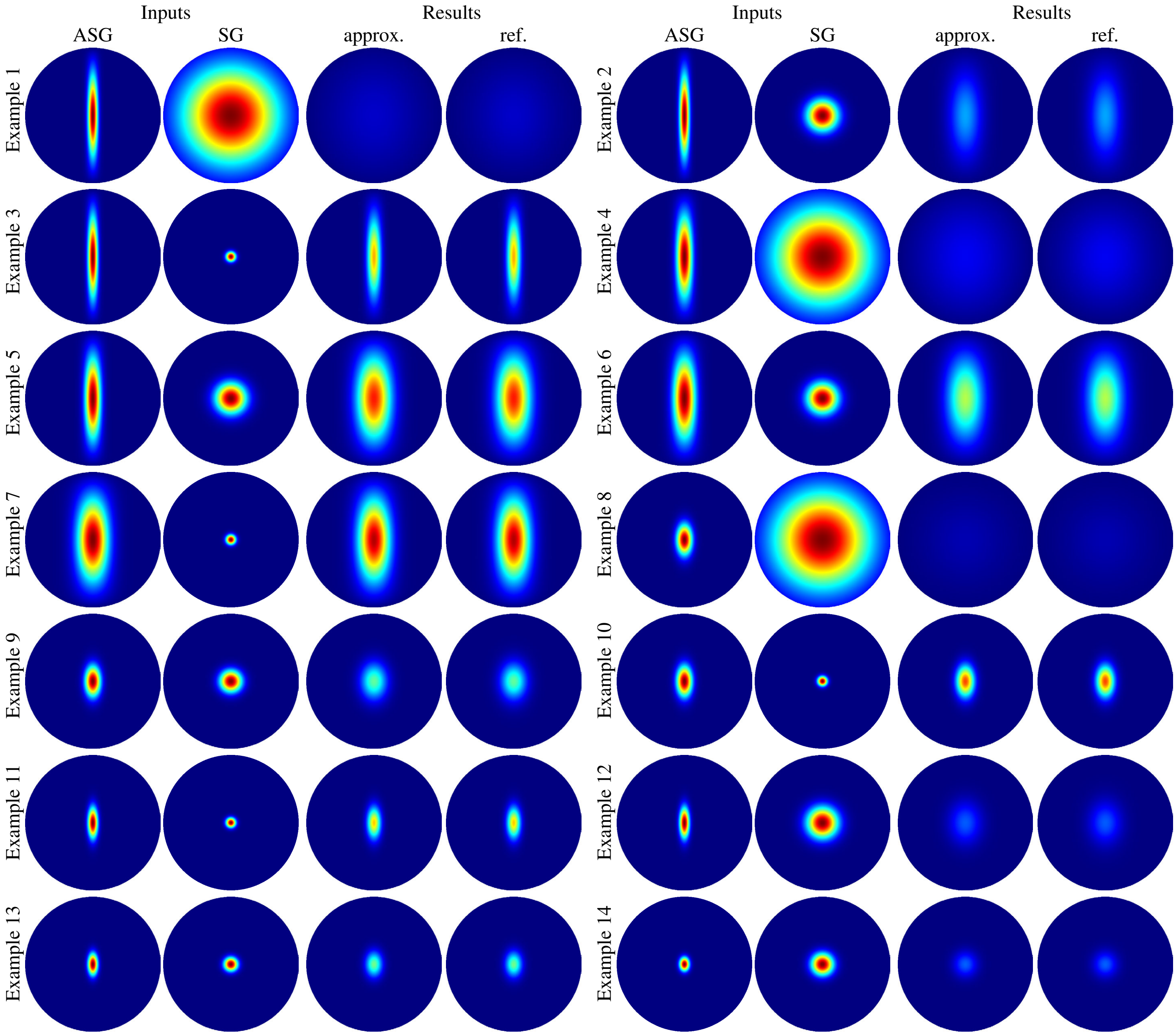


Figure 6: *More Evaluations on ASG convolution operator. The bandwidth parameters of the ASG (λ and μ) and the SG (ν) in all examples are listed below. Example 1:* $\lambda = 100$, $\mu = 1$, $\nu = 1$. *Example 2:* $\lambda = 100$, $\mu = 1$, $\nu = 10$. *Example 3:* $\lambda = 100$, $\mu = 1$, $\nu = 100$. *Example 4:* $\lambda = 40$, $\mu = 1$, $\nu = 1$. *Example 5:* $\lambda = 40$, $\mu = 1$, $\nu = 10$. *Example 6:* $\lambda = 20$, $\mu = 1$, $\nu = 10$. *Example 7:* $\lambda = 10$, $\mu = 1$, $\nu = 100$. *Example 8:* $\lambda = 40$, $\mu = 10$, $\nu = 1$. *Example 9:* $\lambda = 40$, $\mu = 10$, $\nu = 20$. *Example 10:* $\lambda = 40$, $\mu = 10$, $\nu = 100$. *Example 11:* $\lambda = 100$, $\mu = 10$, $\nu = 100$. *Example 12:* $\lambda = 100$, $\mu = 10$, $\nu = 10$. *Example 13:* $\lambda = 100$, $\mu = 20$, $\nu = 50$. *Example 14:* $\lambda = 100$, $\mu = 50$, $\nu = 20$.