A Derivation of the Kirchhoff BRDF Integral for Surface Reflectance

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1 The Scalar Kirchhoff Integral

The following is a derivation of the Kirchhoff scattering integral and how it can be used to estimate a surface's bidirectional reflectance distribution function (BRDF) from its geometry. Other than notation, this derivation is generally similar to standard derivations of the Kirchhoff integral in prior works such as [1, 3, 4, 2].

The goal of this document, besides expressing the equations in our notation, is to familiarize the reader with the steps and assumptions used in deriving the Kirchhoff scattering approximation. The reader is encouraged to see the references for alternate or more rigorous derivations.

1.1 Scalar Approximation

In this note, we are interested for solving for the scattering of light waves from rough surfaces. We will assume that the scattering does not change the frequency of the light (a.k.a. elastic scattering). Thus we can solve for the scattering for each frequency separately. Without loss of generality, let us assume we are dealing with monochromatic light with some wavelength λ.[1]

Light waves consist of oscillating electric and magnetic fields, but for our purposes it is sufficient to only consider the electric field. In a transparent medium (where light is neither scattered nor absorbed), these fields obeys the wave equation:

$$\nabla^2 \vec{\mathbf{E}} - \frac{\eta^2}{c^2} \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0 \tag{1}$$

[1] Technically it is the light's frequency which is fixed instead of its the wavelength, which depends on the local index of refraction η, but it is common usage to refer to light by its wavelength rather than frequency. No confusion should arise as long as the corresponding index of refraction is always known.

i	Imaginary unit for complex numbers, $i^2 = -1$
c	Speed of light in a vacuum
η	Index of refraction of current medium
λ	Wavelength in current medium
k	Wave number $= 2\pi/\lambda$
E_1	Electric field corresponding to the incident light
E_2	Electric field corresponding to the scattered light
$\zeta(x, y)$	Height function for surface
$\mathbf{m}$	Normal to the surface at a point (unit vector)
$\mathbf{n}$	Normal to the average plane of the surface
$A_{\mathcal{S}}^{\perp}$	Projected area of surface perpendicular to $\mathbf{n}$
ψ	Direction towards the light source
ω	Direction towards the camera or sensor
f_{r}	Bidirectional reflectance distribution function (BRDF)

Figure 1: List of some important symbols.

where $\vec{\mathbf{E}}$ is the vector-valued electric field, c is the speed of light, and η is the index of refraction of the medium. This wave equation is derived from Maxwell's equations under the assumptions that the medium is electrically neutral (no charge) and non-conductive (no currents).

Scalar theory offers a simplified framework by removing time and polarization from our equations. First, we assume that the field is a separable product of temporal and spatial terms, so that we can remove time from our equations. For our purposes, we assume the temporal term for the electric fields always has the form: $e^{-i\frac{kc}{\eta}t}$, where $k = 2\pi/\lambda$ is the wave number, and t is time.

Second, let us assume that the orientations of the electric fields are known "a priori" so that we can factor out these known directions, converting from vector fields into a scalar fields. Then we need only solve for the electric field's magnitude and phase, which can be conveniently represented using complex numbers.[2] For light waves, this assumption is equivalent to assuming that the results do not depend on polarization or that we will compute results for each polarization separately. With these assumptions, the wave equation becomes equivalent to the simpler Helmholtz equation:

$$\nabla^2 E + k^2 E = 0 \tag{2}$$

where E is a scalar electric field and k is the wave number (defined as $k = 2\pi/\lambda$).

[2] By convention, the actual electric field is corresponds to the real part of the complex value, but that is not important for these results.

1.2 Scalar Kirchhoff Integral

The Kirchhoff scattering is based on the following integral theorem. Let $\mathcal{S}$ be a closed closed surface bounding some volume $\mathcal{V}$ and let E be a scalar function which obeys the Helmholtz equation inside the volume. If E is known on the boundary (i.e. for points $\vec{\mathbf{s}} \in \mathcal{S}$), then we can compute the value of E for points $\vec{\mathbf{p}}$ inside the volume using the following theorem derived from the mathematical Green's identities:[3]

$$E(\vec{\mathbf{p}}) = \oint_{\mathcal{S}} \left(E(\vec{\mathbf{s}}) \frac{\partial G_{\vec{\mathbf{p}}}(\vec{\mathbf{s}})}{\partial \mathbf{m}(\vec{\mathbf{s}})} - G_{\vec{\mathbf{p}}}(\vec{\mathbf{s}}) \frac{\partial E(\vec{\mathbf{s}})}{\partial \mathbf{m}(\vec{\mathbf{s}})} \right) d\vec{\mathbf{s}} \tag{3}$$

where $G_{\vec{\mathbf{p}}}$ are Green's functions for the Helmholtz equation and $\mathbf{m}$ is the local surface normal of $\mathcal{S}$ (oriented to point inward to the bounded volume).[4] One suitable choice is to use the "free-space" Helmholtz Green's function (which corresponds to field from a radiating point source at $\vec{\mathbf{p}}$):

$$G_{\vec{\mathbf{p}}}(\vec{\mathbf{s}}) = \frac{e^{ik\|\vec{\mathbf{s}}-\vec{\mathbf{p}}\|}}{4\pi \|\vec{\mathbf{s}} - \vec{\mathbf{p}}\|} \tag{4}$$

Since $G_{\vec{\mathbf{p}}}(\vec{\mathbf{s}}) = G_{\vec{\mathbf{s}}}(\vec{\mathbf{p}})$, we can equivalently interpret the integral above as summing the contribution of infinitely many point sources located on the surface $\mathcal{S}$ (similar to Huygen's Principle).

One theoretical problem is that Greens identities assume a complete bounding surface, but in practice we are often interested in scattering from relatively flat (at large scales) surfaces that do not actually bound any volume. The standard solution is to augment our surface $\mathcal{S}$ with a virtual surface $\mathcal{S}'$ such that the combination, $\mathcal{S} \cup \mathcal{S}'$, bounds a large volume (which always contains $\vec{\mathbf{p}}$). This virtual surface $\mathcal{S}'$ is constructed in such a way, that we can argue that its contribution to the integral is negligible and thus we need only actually compute the integral over our original surface $\mathcal{S}$. For example the surface $\mathcal{S}'$ can be constructed such that the most of it is too far away from $\vec{\mathbf{p}}$ to affect it significantly, while the closer portions are designed to lie in regions where the expect the electric field to be zero, such as in the shadow of an opaque surface.

1.3 Relation to BRDF

Our goal to is estimate how light scattered from a surface $\mathcal{S}$ when it is illuminated by some incident light, which we represent as an incident electric field E_1. This incident field

[3] In Kirchhoff theory, this equation is often written with a factor of $1/4\pi$ in front (e.g., [1]), but instead we incorporate this factor inside $G_{,}$ to be consistent with the standard definition for Green's functions (e.g., as in [3]).

[4] This integral depends on both E and its normal derivative on the surface. In theory, this is over-determined and just knowing one of these would be sufficient. Choosing Green's functions such that their values or their normal derivatives are always zero on the surface, would remove one of the terms. In practice however, finding such specialized Green's functions is only feasible for trivial surfaces.

interacts locally with the surface (e.g., causing electrons to move), thereby creating an additional induced electric field E_2 on the surface. This induced field then propagates outward as scattered light. Light energy is proportional to the squared magnitude of its electric field. Thus by estimating the magnitude of E_2 in various directions and for various incident lighting configurations, we can characterize how a surface scatters light.

Let us assume the incident light is a columnated beam arriving from a direction ψ. Since we are not concerned here with the beam's size, and only care about relative phase and energy differences, we will model it as an infinite plane wave traveling in direction $-\psi$ with unit magnitude as:

$$E_1(\vec{\mathbf{p}}) = e^{-ik(\psi \cdot \vec{\mathbf{p}})} \tag{5}$$

Bidirectional reflectance distribution functions (BRDF) are used to describe how a surface scatters light. If we illuminate a surface with light from a direction ψ and measure how much light is scattered in direction ω, then the BRDF is defined as the ratio between the scattered radiance and the incident irradiance. Let us assume our surface is flat, at least at large scales, with an average surface normal $\mathbf{n}$, and projected area $A_{\mathcal{S}}^{\perp}$ relative to $\mathbf{n}$. Then the the incident irradiance is $|\psi \cdot \mathbf{n}|\,|E_1|^2$. We can estimate the scattered radiance by measuring the light energy reaching a far away point divided by the solid angle of the surface as seen from that point. Let $\vec{\mathbf{p}} = r\omega$ for large distances r, then the solid angle subtended by the surface as seen from $\vec{\mathbf{p}}$ is $A_{\mathcal{S}}^{\perp}\,|\omega \cdot \mathbf{n}|\,/r^2$. To get the scattered radiance in direction ω, we take the limit as r increases. Putting this together we get the following expression for the BRDF:

$$f_{\mathrm{r}}(\psi,\omega) = \lim_{r\to\infty} \frac{r^2}{A_{\mathcal{S}}^{\perp}\,|\psi \cdot \mathbf{n}|\,|\omega \cdot \mathbf{n}|} \frac{|E_2(r\omega)|^2}{|E_1|^2} \tag{6}$$

Next, we want to use scalar Kirchhoff integrals to evaluate E_2 in this expression. To do this we need to assume some boundary conditions for E_2 on the surface and use the appropriate Green's functions in the limit of large r.

1.4 Boundary Conditions

Kirchhoff scattering theory makes the simplifying assumption that the induced field on the surface is linearly related to the incident field by a reflection coefficient R so that:

$$E_2(\vec{\mathbf{s}}) \;=\; R(\vec{\mathbf{s}},\psi)E_1(\vec{\mathbf{s}}) \;=\; R(\vec{\mathbf{s}},\psi)e^{-ik(\psi \cdot \vec{\mathbf{s}})} \quad \text{for} \;\; \vec{\mathbf{s}} \in \mathcal{S} \tag{7}$$

There are several assumptions implicit in this approximation. The incident field is assumed to reach the entire surface unimpeded (i.e. no shadowing or masking). The induced field on the surface is also assumed to be unaffected by the induced field from other parts of the surface (i.e. no multiple scattering). These are similar to some of the assumptions used in microfacet scattering models in geometric optics.

To use the Kirchhoff scattering integral we also need an approximation to the normal derivative of the induced field at the surface. We will use the following assumption:

$$\frac{\partial E_2(\vec{\mathbf{s}})}{\partial \mathbf{m}(\vec{\mathbf{s}})} = -R(\vec{\mathbf{s}},\psi)\frac{\partial E_1(\vec{\mathbf{s}})}{\partial \mathbf{m}(\vec{\mathbf{s}})} = ikR(\vec{\mathbf{s}},\psi)\,(\psi\cdot\mathbf{m}(\vec{\mathbf{s}}))\,e^{-ik(\psi\cdot\vec{\mathbf{s}})} \quad \text{for} \quad \vec{\mathbf{s}}\in\mathcal{S} \tag{8}$$

Beckmann [1] offers two arguments for this choice. One is based on analyzing the expected behavior in the case of a locally flat surface. In this case the scattered field is also a plane wave, traveling in the mirror reflection direction with a magnitude R (which can be computed from Fresnel's equations in this case). Differentiating this reflected plane wave in the normal direction matches the approximation given above.

1.5 Far-field Green's Function

Our BRDF equation contains the limit $r\to\infty$. We can handle this by defining a far-field Green's function.[5] Setting $\vec{\mathbf{p}} = r\omega$, we can use a Taylor series approximation to get:

$$\|\vec{\mathbf{s}} - r\omega\| \approx r - (\vec{\mathbf{s}}\cdot\omega) + O(\frac{1}{r}) \tag{9}$$

then substituting this into our Green's functions and keeping only the non-vanishing terms in the limit of large r, we get:

$$G_\omega(\vec{\mathbf{s}}) \equiv \lim_{r\to\infty} G_{r\omega}(\vec{\mathbf{s}}) \approx \frac{e^{ikr}e^{-ik\left(\vec{\mathbf{s}}\cdot\omega\right)}}{4\pi r} \approx \frac{e^{-ik\left(\vec{\mathbf{s}}\cdot\omega\right)}}{4\pi r} \tag{10}$$

where in the last part we drop the e^{ikr} as its an overall phase shift that will not affect our results. Equivalently we could require that r to go infinity in a way such that kr is always a multiple of 2π (so that $e^{ikr}=1$). Using a similar derivation we can get:

$$\frac{\partial G_\omega(\vec{\mathbf{s}})}{\partial \mathbf{m}(\vec{\mathbf{s}})} \approx \frac{-ik\,(\omega\cdot\mathbf{m}(\vec{\mathbf{s}}))\,e^{-ik\left(\vec{\mathbf{s}}\cdot\omega\right)}}{4\pi r} \tag{11}$$

Combining these with the scalar Kirchhoff integral gives:

$$E_2(r\omega) = \frac{-ik}{4\pi r}\oint_{\mathcal{S}} R(\vec{\mathbf{s}},\psi)\,((\psi+\omega)\cdot\mathbf{m}(\vec{\mathbf{s}}))\,e^{-ik\left((\psi+\omega)\cdot\vec{\mathbf{s}}\right)}\,d\vec{\mathbf{s}} \tag{12}$$

To simplify the notation, let us define a vector $\vec{\mathbf{q}}$ as:

$$\vec{\mathbf{q}} \equiv k\,(\psi+\omega) = \frac{2\pi\,(\psi+\omega)}{\lambda} \tag{13}$$

Then we get the following expression for the BRDF:

$$f_{\mathrm{r}}(\psi,\omega) = \frac{1}{16\pi^2 A_{\mathcal{S}}^{\perp}\,|\psi\cdot\mathbf{n}|\,|\omega\cdot\mathbf{n}|}\left|\oint_{\mathcal{S}} R(\vec{\mathbf{s}},\psi)\,(\vec{\mathbf{q}}\cdot\mathbf{m}(\vec{\mathbf{s}}))\,e^{-i\left(\vec{\mathbf{q}}\cdot\vec{\mathbf{s}}\right)}\,d\vec{\mathbf{s}}\right|^2 \tag{14}$$

[5] We provide only an informal derivation here, without proving that moving the limit inside the integral is allowed in this case.

1.6 Constant R Approximation

Although this expression is fairly compact, the integral is difficult to analyze so often further simplifying assumptions are used to make it more tractable. One common approximation is the assume that R is constant (or at least independent of position) and can thus be moved outside of the integral. We can then write it as:

$$f_{\mathrm{r}}(\psi,\omega) = \frac{|R|^2\, D_{\mathrm{K}}(\vec{\mathbf{q}})}{4\,|\psi\cdot\mathbf{n}|\,|\omega\cdot\mathbf{n}|} \tag{15}$$

where we have defined a new function D_{K}, which we call the Kirchhoff distribution, as:

$$D_{\mathrm{K}}(\vec{\mathbf{q}}) = \frac{1}{4\pi^2 A_{\mathcal{S}}^{\perp}} \left| \oint_{\mathcal{S}} (\vec{\mathbf{q}}\cdot\mathbf{m}(\vec{\mathbf{s}}))\, e^{-i(\vec{\mathbf{q}}\cdot\vec{\mathbf{s}})}\, d\vec{\mathbf{s}} \right|^2 \tag{16}$$

This is the form of the Kirchhoff scattering integral we use in the accompanying paper and completes our derivation. □

1.7 Relation to Prior Derivations

While this result is mathematically equivalent to prior derivations in standard texts, there are a few differences besides the notation. None of the prior derivations we have seen define the D_{K} function or expressed the result in exactly this form, which we think helps elucidate the similarity and differences between Kirchhoff and microfacet theories and their scattering predictions.

The integral form we use depends on both the surface positions $\vec{\mathbf{s}}$, and surface normals $\mathbf{m}(\vec{\mathbf{s}})$. However, the dependence on surface normals is often considered inconvenient when creating analytic approximations to this integral, and most derivations include an additional approximation to remove it. One method is to use integration by parts to convert the integral into a sum of a 2D integral that does not include the surface normal and a 1D integral around the boundary of the surface. Then typically some condition is assumed such that the 1D integral is zero or has negligible contribution and can thus be discarded. For example the surface may be assumed to be very large (or even infinite) or that the boundary lies within a particular 2D plane. We did not apply this transformation here both because we do not need it and because we did not want to have to add the additional assumptions it requires. We will estimate the integral numerically from explicit surface data, so computing the corresponding surface normals is not difficult.

2 Similarity to Microfacet BRDFs

Microfacet theory, which is derived assuming geometric optics, predicts BRDFs that have strikingly similar form to our Kirchhoff expresssion. A microfacet BRDF is given by:

$$f_{\mathrm{r,M}}(\psi,\omega) = \frac{F(\psi\cdot\mathbf{h})\,G(\psi,\omega)\,D_{\mathrm{M}}(\mathbf{h})}{4\,|\psi\cdot\mathbf{n}|\,|\omega\cdot\mathbf{n}|} \tag{17}$$

where F is the Fresnel reflectance, G is the shadowing/masking term, and D_M is the microfacet normal distribution function. These functions depend on the half direction (also called the half vector) defined as:

$$\mathbf{h} \equiv \frac{\psi + \omega}{\|\psi + \omega\|} = \frac{\vec{\mathbf{q}}}{\|\vec{\mathbf{q}}\|} \tag{18}$$

The microfacet and Kirchhoff BRDF equations are very similar in form but with a few crucial differences. The Fresnel term F is analogous to the $|R|^2$ and indeed R is often computed from the same Fresnel equations. Kirchhoff has no G term because it neglects the effects of shadowing and masking. However adding such a term can improve its energy conservation especially for near grazing angles. The D_M in microfacet models is roughly analogous to the D_K term, except that it is a 2D function instead of 3D (its argument is a unit vector instead of a general vector). However if cases where the D_K is only weakly dependent on the length of $\vec{\mathbf{q}}$, these two different model can predict very similar BRDFs.

This similarity is perhaps not surprising as its well-known that wave optics is a more general theory and matches geometrics optics predictions under the conditions where geometric optics is accurate. Both Beckmann [1] and Stam [4] have previously derived a microfacet-style BRDF as an approximation to the Kirchhoff predictions for a particular class of surfaces under suitable conditions. Our formulation provides a more general expression of their similarity and a more general condition under which they produce equivalent predictions, namely whenever the value of D_K is roughly independent of the length of $\vec{\mathbf{q}}$. We know this condition holds for many but not all surfaces, however it is currently an open question exactly what class of surfaces it holds for.

References

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